Impact of the pion mass on nonpower expansion for QCD observables

A.V. Nesterenko[ab] and J. Papavassiliou[a]

[a]Departamento de Física Teórica and IFIC, Centro Mixto, Universidad de Valencia–CSIC, E-46100, Burjassot, Valencia, Spain

[b]Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, 141980, Russian Federation

A new set of functions, which form a basis of the massive nonpower expansion for physical observables, is presented in the framework of the analytic approach to QCD at the four-loop level. The effects due to the π meson mass are taken into account by employing the dispersion relation for the Adler function. The nonvanishing pion mass substantially modifies the functional expansion at low energies. Specifically, the spacelike functions are affected by the mass of the π meson in the infrared domain below few GeV, whereas the timelike functions acquire characteristic plateaulike behavior below the two–pion threshold. At the same time, all the appealing features of the massless nonpower expansion persist in the considered case of the nonvanishing pion mass.

The renormalization group (RG) method plays a key role in the framework of the Quantum Field Theory (QFT) and its applications. Indeed, one is able to handle reliably the strong interaction processes at high energies by employing this method together with perturbative calculations. However, such perturbative solutions to the RG equation possess unphysical singularities in the infrared domain, a fact that contradicts the general principles of the local QFT, and significantly complicates the theoretical description and interpretation of the intermediate- and low-energy experimental data. Nevertheless, an effective way to overcome these difficulties is to complement the perturbative results with a proper nonperturbative insight into the infrared hadron dynamics.

One of the sources of the nonperturbative information about the strong interaction processes is the dispersion relations. The idea of employing the latter together with perturbation theory forms the underlying concept of the so-called analytic approach to QFT, which was first proposed in the framework of Quantum Electrodynamics [1]. Recently, this approach has been extended to Quantum Chromodynamics (QCD) [2] and applied to the "analytization" of the perturbative power series for the QCD observables [3,4,5]. The term analytization means the restoring of the correct analytic properties in the kinematic variable of a quantity under consideration by making use of the Källén–Lehmann integral representation (positive q^2 corresponds to a spacelike momentum transfer hereinafter)

$$\Big\{F(q^2)\Big\}_{\rm an} = \int_0^\infty \frac{\varrho(\sigma)}{\sigma + q^2}\, d\sigma \tag{1}$$

with the spectral function defined by the initial (perturbative) expression for the quantity at hand:

$$\varrho(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[F(-\sigma - i\varepsilon) - F(-\sigma + i\varepsilon)\Big]. \tag{2}$$

However, there are several ways to embody the analyticity requirement into the RG formalism, that eventually has given rise to different models for the analytic running coupling.

Thus, in the original model due to Shirkov and Solovtsov [2] the analyticity requirement (1) is imposed on the perturbative running coupling itself. At the one-loop level this leads to

$$\alpha_{\rm ss}^{(1)}(q^2) = \frac{4\pi}{\beta_0}\left(\frac{1}{\ln z} + \frac{1}{1-z}\right), \qquad z = \frac{q^2}{\Lambda^2}, \tag{3}$$

whereas at the higher loop levels the integral representation of the Källén–Lehmann type

$$\alpha(q^2) = \frac{4\pi}{\beta_0} \int_0^\infty \frac{\rho(\sigma)}{\sigma + q^2}\, d\sigma \tag{4}$$

holds for this invariant charge. Ultimately, the prescription [2] results in the infrared finite limiting value for the running coupling (see papers [3,4,5] and references therein for the details).

Another way to incorporate the analyticity condition into the RG formalism is to impose the analyticity requirement (1) on the perturbative approximation of the β function with subsequent solution of the corresponding RG equation [6]. At the one-loop level this leads to

$$\alpha_{\mathrm{an}}^{(1)}(q^2) = \frac{4\pi}{\beta_0}\frac{z-1}{z\,\ln z}, \tag{5}$$

whereas at the higher loop levels the running coupling at hand can be represented in the form of the Källén–Lehmann integral (4) as well. Here the invariant charge possesses the infrared enhancement, which plays an essential role in applications of this model to the study of the quark confinement [6] and the chiral symmetry breaking [7]. It is of particular interest to mention that the explicit one-loop form of the analytic running coupling (5) has recently been rediscovered [8], proceeding from entirely different motivations.

The dispersion relations also play an important role for the congruous description of the hadron dynamics in the spacelike and timelike regions. In particular, it has been argued that the dispersion relation for the Adler function [9]

$$D(q^2, m_\pi^2) = q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+q^2)^2}\, ds \tag{6}$$

provides a firm ground for comparing the perturbative results for $D(q^2, m_\pi^2)$ with the measurable ratio $R(s)$ of the e^+e^- annihilation into hadrons. Indeed, one can continue an explicit expression for the Adler function into the timelike domain by making use of the inverse relation

$$R(s, m_\pi^2) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta, m_\pi^2)\, \frac{d\zeta}{\zeta}, \tag{7}$$

where the integration contour lies in the region of the analyticity of the integrand (see Refs. [10,11]).

The only information about the Adler function (6) available from the perturbation theory is its behavior in the asymptotical ultraviolet region $q^2 \to \infty$. Specifically, at the ℓ-loop level

$$D(q^2) \simeq 1 + \sum\nolimits_{j=1}^{\ell} d_j \left[a_{\mathrm{s}}^{(\ell)}(q^2)\right]^j, \tag{8}$$

where the overall factor $N_{\mathrm{c}} \sum_f Q_f^2$ is omitted throughout, $a_{\mathrm{s}}(q^2) = \alpha_{\mathrm{s}}(q^2)\beta_0/(4\pi)$ is the perturbative "couplant", $\beta_0 = 11 - 2n_{\mathrm{f}}/3$, n_{f} is the number of active flavors, and $d_1 = 4/\beta_0$, $d_2 \simeq (4/\beta_0)^2\,(1.99 - 0.12\,n_{\mathrm{f}})$, see Refs. [12,13] for the details. At the same time, the dispersion relation (6) implies that $D(q^2, m_\pi^2)$ is the analytic function in the complex q^2-plane with the only cut along the negative semiaxis of real q^2 beginning at the two–pion threshold. Thus, the perturbative approximation (8) violates this condition due to unphysical singularities of $\alpha_{\mathrm{s}}(q^2)$. Nonetheless, this disagreement can be avoided within the analytic approach to QCD.

In particular, in the framework of the so–called analytic perturbation theory (APT) [3,4,5] the pion mass was ignored in dispersion relation (6), and the analyticity requirement of the form (1) has been imposed on the perturbative approximation (8). Since both real and imaginary parts of the strong running coupling contribute to the relevant spectral density, eventually this led to the nonpower expansion for the Adler function:

$$D(q^2) = 1 + \sum\nolimits_{j=1}^{\ell} d_j\, \mathcal{A}_{\mathrm{SL,\,j}}^{(\ell)}(q^2) \tag{9}$$

(subscript "SL" stands for "spacelike"), where

$$\mathcal{A}_{\mathrm{SL,\,j}}^{(\ell)}(q^2) = \int_0^{\infty} \frac{\varrho_{\mathrm{j}}^{(\ell)}(\sigma)}{\sigma + q^2}\, d\sigma \tag{10}$$

and

$$\varrho_{\mathrm{j}}^{(\ell)}(\sigma) = \frac{1}{\pi} \lim_{\varepsilon \to 0_+} \mathrm{Im}\left\{\left[a_{\mathrm{s}}^{(\ell)}(-\sigma - i\varepsilon)\right]^j\right\}. \tag{11}$$

In turn, the continuation of $D(q^2)$ (9) into timelike domain (7) can also be represented in a form of the nonpower functional expansion:

$$R(s) = 1 + \sum\nolimits_{j=1}^{\ell} d_j\, \mathcal{A}_{\mathrm{TL,\,j}}^{(\ell)}(s) \tag{12}$$

(subscript "TL" stands for "timelike" here), with

$$\mathcal{A}_{\mathrm{TL,\,j}}^{(\ell)}(s) = \int_s^{\infty} \varrho_{\mathrm{j}}^{(\ell)}(\sigma)\, \frac{d\sigma}{\sigma}. \tag{13}$$

Since the functions (13) automatically take into account the so-called π^2–terms, the expansion coefficients d_j in Eqs. (9) and (12) are identical. The first–order expansion functions ($j = 1$) correspond to the ℓ-loop running couplings in spacelike (10) and timelike (13) domains, whereas the

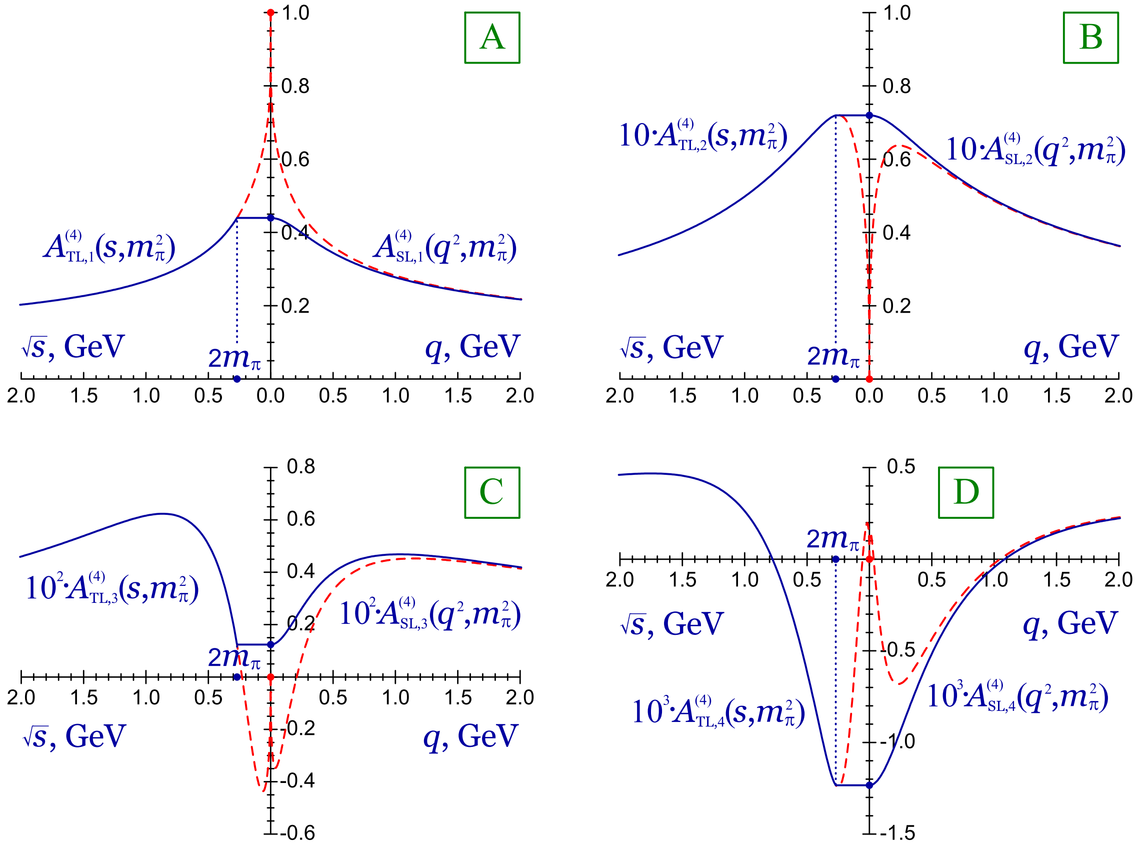

Figure 1. The four-loop nonpower expansion functions in spacelike $[A^{(4)}_{\rm SL,\,j}(q^2, m_\pi^2)$, $q^2 > 0$, Eq. (16)] and timelike $[A^{(4)}_{\rm TL,\,j}(s, m_\pi^2)$, $s = -q^2 > 0$, Eq. (18)] regions. The plots A, B, C, and D correspond to the first-, second-, third-, and fourth-order expansion functions ($j = 1, 2, 3, 4$), respectively. The case of the nonvanishing mass of the π meson is denoted by solid curves, whereas the dashed curves show the limit of the massless pion [see Eqs. (10) and (13)]. The values of parameters are: $\Lambda = 500\,\mathrm{MeV}$, $n_{\rm f} = 2$.

higher–order functions ($2 \le j \le \ell$) play the role of their effective powers. The sets of those functions form the ℓ-loop bases of the nonpower expansions. Remarkably, the spacelike functions (10) deviate from the perturbative expansion basis at rather high energies. Thus, $A^{(4)}_{\rm SL,\,1}(q^2)$ differs by 20 % from $a_{\rm s}^{(4)}(q^2)$ at $q \simeq 2\,\mathrm{GeV}$, whereas the difference between $A^{(4)}_{\rm SL,\,4}(q^2)$ and $[a_{\rm s}^{(4)}(q^2)]^4$ is 40 % at $q \simeq 10\,\mathrm{GeV}$ (see also Refs. [3,4,5] for the details).

In fact, the effects due to the masses of the light hadrons can be safely neglected only when one handles the strong interaction processes at high energies. However, in the intermediate- and low-energy regions such mass effects become substantial. So, for the case of the nonvanishing pion mass, one can bring the perturbative expansion (8) in conformity with the dispersion relation (6) by requiring the former to satisfy the integral representation of the form (see Ref. [14])

$$D(q^2, m_\pi^2) = \int_{4m_\pi^2}^{\infty} \frac{\varkappa(\sigma)}{\sigma + q^2}\, d\sigma. \qquad (14)$$

Ultimately, this also leads to the nonpower expansions for the Adler function and $R_{e^+e^-}$ ratio:

$$D(q^2, m_\pi^2) = \frac{q^2}{q^2 + 4m_\pi^2} + \sum_{j=1}^{\ell} d_j A^{(\ell)}_{\rm SL,\,j}(q^2, m_\pi^2), \qquad (15)$$

where

$$A^{(\ell)}_{\mathrm{SL,\,j}}(q^2, m_\pi^2) = \int_{4m_\pi^2}^{\infty} \frac{\varrho_{\mathrm{j}}^{(\ell)}(\sigma)}{\sigma + q^2}\, d\sigma \qquad (16)$$

[the spectral function (11) is adopted herein]. It is worth noting that $A^{(\ell)}_{\mathrm{SL,\,1}}(q^2, m_\pi^2)$ is a process–dependent quantity, which can be identified with the QCD invariant charge at high energies only, where the influence of the pion mass on Eq. (16) is negligible. Then, the continuation of $D(q^2, m_\pi^2)$ [see Eq. (15)] into timelike domain (7) reads

$$R(s, m_\pi^2) = \theta(s - 4m_\pi^2) + \sum_{j=1}^{\ell} d_j A^{(\ell)}_{\mathrm{TL,\,j}}(s, m_\pi^2), \quad (17)$$

where $\theta(x)$ is the Heaviside step function and

$$A^{(\ell)}_{\mathrm{TL,\,j}}(s, m_\pi^2) = \int_{s}^{\infty} \theta(\sigma - 4m_\pi^2)\, \varrho_{\mathrm{j}}^{(\ell)}(\sigma) \frac{d\sigma}{\sigma}. \qquad (18)$$

It is worth emphasizing here that the main impact of the mass of the π meson on Eqs. (15) and (17) is twofold; not only the strong corrections, but also the parton model predictions are modified at low energies (see Ref. [15] for the details).

The four-loop massive nonpower expansion functions (16) and (18) are presented in Figure 1. It turns out that all the appealing features of the massless APT [3,4,5] persist in the considered case of the nonvanishing pion mass. Specifically, the functions (16) and (18) have no unphysical singularities and contain no additional parameters. The timelike expansion functions (18) automatically take into account the π^2–terms. The higher–order functions are suppressed with respect to the preceding ones, a fact that ultimately leads to the higher loop and scheme stability of outcoming results. It is worth noting that the spacelike expansion functions (16) are influenced by the pion mass in the infrared domain below few GeV, whereas the timelike expansion functions (18) are affected by the mass of the π meson only below the two–pion threshold, where they acquire characteristic plateaulike behavior (see Figure 1).

The impact of the effects due to the nonvanishing mass of the π meson on the model (5) (see Ref. [6]) and on the processing the experimental data on the inclusive τ lepton decay has been discussed in detail in Ref. [14].

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REFERENCES

1. P.J. Redmond, Phys. Rev. **112**, 1404 (1958); N.N. Bogoliubov, A.A. Logunov, and D.V. Shirkov, Sov. Phys. JETP **37**, 574 (1960).
2. D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett. **79**, 1209 (1997); hep-ph/9604363.
3. I.L. Solovtsov and D.V. Shirkov, Theor. Math. Phys. **120**, 1220 (1999); D.V. Shirkov, *ibid.* **127**, 409 (2001).
4. K.A. Milton and I.L. Solovtsov, Phys. Rev. D **55**, 5295 (1997); **59**, 107701 (1999).
5. D.V. Shirkov, Eur. Phys. J. C **22**, 331 (2001); hep-ph/0408272.
6. A.V. Nesterenko, Phys. Rev. D **62**, 094028 (2000); **64**, 116009 (2001); Int. J. Mod. Phys. A **18**, 5475 (2003).
7. A.C. Aguilar, A.V. Nesterenko, and J. Papavassiliou, J. Phys. G **31**, 997 (2005).
8. F. Schrempp, J. Phys. G **28**, 915 (2002).
9. S.L. Adler, Phys. Rev. D **10**, 3714 (1974).
10. A.V. Radyushkin, JINR report No. 2-82-159 (1982); JINR Rapid Comm. **4**, 9 (1996); hep-ph/9907228.
11. N.V. Krasnikov and A.A. Pivovarov, Phys. Lett. B **116**, 168 (1982).
12. S.G. Gorishny, A.L. Kataev, and S.A. Larin, Phys. Lett. B **259**, 144 (1991).
13. L.R. Surguladze and M.A. Samuel, Phys. Rev. Lett. **66**, 560 (1991); **66**, 2416(E) (1991).
14. A.V. Nesterenko and J. Papavassiliou, Phys. Rev. D **71**, 016009 (2005); hep-ph/0409220.
15. A.V. Nesterenko and J. Papavassiliou, in preparation.